Unbiased estimation of Weibull parameters with the linear regression method

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Abstract

Monte Carlo simulations were used to search for the probability estimator for the unbiased estimate of the Weibull parameters in the linear regression method. Compared with commonly-used probability estimators, the estimator proposed gives a more accurate estimation of the Weibull modulus and the same estimation precision of the scale parameter. It is found that the estimator proposed is more conservative than the estimator $P_i = (i - 0.5)/n$ recommended by previous authors, and hence results in a higher safety in reliability predictions. The unbiased properties of the estimated Weibull parameters were validated with actual experimental data. It is also concluded that the estimated Weibull modulus from actual experimental data is more dispersive than that from Monte Carlo simulation, which arises from the fact that the strength data from actual experiments does not perfectly follow the Weibull statistics.
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1. Introduction

Weibull statistics has been commonly used to characterize the statistical variation in the fracture strength of brittle materials such as ceramics, glasses and solid catalysts.[1–4] It is based on a "weakest link theory", which means that the most serious flaw in the material will control the strength, like a chain breaking if the weakest link fails. The most serious flaw is not necessarily the largest one because its severity also relies on its location and orientation. In other words, the flaw subjected to the highest stress intensity factor will be strength controlling.

Using Weibull's two-parameter distribution, the cumulative probability of failure P at or below a stress σ is represented by[1,2,5]

$$P = 1 - \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^m\right] \tag{1}$$

where m and σ_0 are the Weibull modulus and the scale parameter, respectively. The Weibull modulus m, also called the shape parameter, represents the scatter in the fracture strength. A higher m leads to a steeper distribution function and thus a lower dispersion of the fracture strength. The scale parameter σ_0 corresponding to the fracture stress with a failure probability of 63.2% is closely related to the mean strength of the distribution, $\bar{\sigma}$.[1,5]

$$\bar{\sigma} = \sigma_0 \Gamma\left(1 + \frac{1}{m}\right) \tag{2}$$

where Γ is the gamma function. For the Weibull modulus of 5–20, a typical range for technical ceramics,[5] $\Gamma(1 + 1/m)$ takes values between 0.9 and 1.

There are several methods available in the literature,[5–14] for the determination of the Weibull distribution parameters from a set of experimentally measured fracture stresses. It has been shown that the maximum likelihood (ML) method leads to the highest estimation precision of the Weibull modulus, which has been recommended by previous authors.[1,5,7]

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However, the most widely used may be the linear regression (LR) method due to its simplicity. Moreover, the ML method results more often in an overestimation of the Weibull modulus than underestimation, and hence results in a lower safety than the LR method in reliability prediction.[10,14] From an engineering point of view, the LR method is, therefore, to be preferred.

In the LR method, the measured fracture stresses are ranked in ascending order and then a probability of failure P_i, is assigned to each stress σ_i. Since the true value of P_i is unknown, a prescribed estimator has to be used. The following four expressions are often applied to define the probability estimator.[8–12]

$$P_i = \frac{i - 0.5}{n} \tag{3a}$$

$$P_i = \frac{i}{n + 1} \tag{3b}$$

$$P_i = \frac{i - 0.3}{n + 0.4} \tag{3c}$$

$$P_i = \frac{i - 3/8}{n + 1/4} \tag{3d}$$

where P_i is the probability of failure for the ith ranked stress datum, and n is the sample size.

By taking the logarithm twice, Eq. (1) can be rewritten in a linear form.

$$\ln\ln\left(\frac{1}{1 - P}\right) = m\ln\sigma - m\ln\sigma_0 \tag{4}$$

The Weibull modulus can thus be obtained directly from the slope term in Eq. (4) and the scale parameter can be deduced from the intercept term.

However, the estimators of the Weibull modulus are always biased for both the ML and LR methods.[5–7,15,16] In most cases the bias increases rapidly as the sample size decreases. In the LR method, different probability estimators also lead to different biases of the estimated Weibull modulus. It has been shown that Eq. (3b) gives the largest-biased estimate of the Weibull modulus, while Eq. (3a) results in the least bias for $n \geq 20$.[9–12] To overcome this shortcoming, several authors proposed the use of a correction factor to adjust the bias of the estimated Weibull modulus.[9,15,16] However, both analytical analyses and numerical calculations revealed that each set of strength data gives the statistically correct Weibull parameters and that the bias arises only from the method of adding the parameters, if one tries to obtain a mean value from a number of sets of strength data.[17] In practice, when only one set of strength data is available, the correction factor should not be applied.

The probability estimator has a significant effect on the bias of the estimated Weibull modulus in the LR method. The objective of this paper is to try to find appropriate expressions to define the probability estimator, which leads to the unbiased estimation of the Weibull parameters. A Monte Carlo simulation was used for this purpose.

2. Monte Carlo simulation

From Eqs. (3a)–(3d), it is shown that the probability estimator should have the following general expression.

$$P_i = \frac{i - \alpha}{n + \beta} \tag{5}$$

By varying the values of α and β, large numbers of functions can be obtained. Clearly, the α-value is required to be less than unity; otherwise a negative value of the probability of failure in case of $i = 1$ arises. Due to the requirement of statistical inference, the β-value is also necessary to be very small, usually not larger than unity, especially for $n \leq 50$.[14] The disadvantage of a large β-value is that a poor precision of the estimated scale parameter takes place.[13] Therefore, the numeric areas of the α and β-values are generally $0 \leq \alpha < 1$ and $0 \leq \beta \leq 1$, respectively.

If α and β are restricted to two decimals, a total of 101 values are obtained for β and 100 values for α due to $\alpha < 1$. And then $\alpha = 0.999$ is added. As a result, 101^2 possible combinations of the α and β-values are produced. In order to search for the combination leading to an unbiased estimate of the Weibull modulus, a Monte Carlo simulation was used, as shown in Fig. 1.

Eq. (1) can be rewritten as

$$\sigma = \sigma_0\left[\ln\left(\frac{1}{1 - P}\right)\right]^{1/m} \tag{6}$$

If we consider a large "specimen" population with prescribed m and σ_0 values, i.e. m_{true} and $\sigma_{0,\text{true}}$, random strength data can be obtained from Eq. (6) provided random numbers between 0 and 1 are substituted for the probability of failure P. For the sake of convenience, we let $m_{\text{true}} = 10$ and $\sigma_{0,\text{true}} = 1$ throughout this study. A computer program was written, which used a sample of random numbers to obtain strength values $\sigma_1, \sigma_2, \ldots, \sigma_i, \ldots, \sigma_n$. This set of strength values, regarded as a fictitious sample, was ranked in ascending order and the probabilities of failure were calculated from Eq. (5) with each combination of the α and β-values. The sample was then analyzed with the LR method to give the estimated value of the Weibull modulus. This procedure was repeated 10,000 times. Consequently, a total of 10,000 samples were generated and 10,000 estimated values of the Weibull modulus were obtained for each combination of the α and β-values. Then the mean value $\bar{m}$, standard deviation S_{m} and coefficient of variation CV_{m} of these moduli were computed from

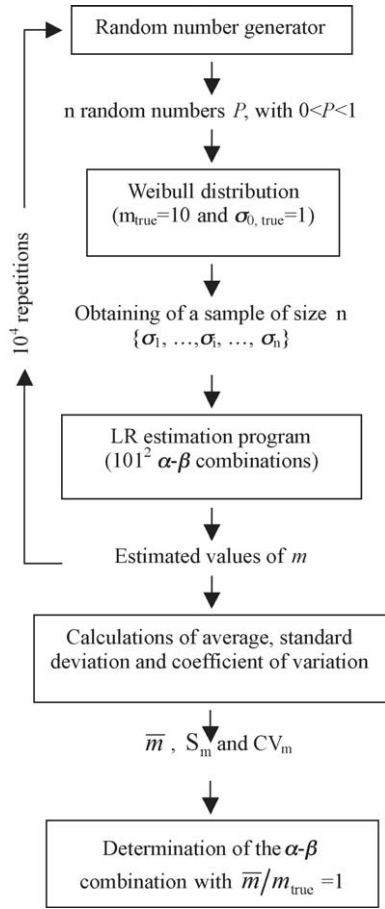$$\bar{m} = \sum_{j=1}^{10^4} \frac{m_j}{10^4} \tag{7}$$

Fig. 1. Schematic flow diagram of the Monte Carlo simulation.

$$S_m{}^2 = \sum_{j=1}^{10^4} \frac{(m_j - \bar{m})^2}{10^4 - 1} \qquad (8)$$

$$CV_m = \frac{S_m}{\bar{m}} \qquad (9)$$

where m_j is the estimated Weibull modulus of the jth sample. Clearly, the combination of the α and β-values, which makes $\bar{m}/m_{true}$ be equal to unity, leads to an unbiased estimate of the Weibull modulus.

For illustration of the effect of the sample size, the generated random samples were of size $n = 10$, 15, 20, 25, 30, 35, 40, 45 and 50. The combination of the α and β-values leading to an unbiased estimate of the Weibull modulus was determined for each sample size.

It should be pointed out that, although the above simulation was carried out for the arbitrarily chosen values of $m_{true} = 10$ and $\sigma_{0,true} = 1$ its results are valid for any value of m_{true} and $\sigma_{0,true}$, for previous studies have shown that the values of $\bar{m}/m_{true}$ and S_m/m_{true}, and the distribution of the estimated m/m_{true} are independent of the prescribed values of m_{true} and $\sigma_{0,true}$ for the LR method.[5,6,9,10]

3. Experimental

A sample of alumina agglomerates, often used as catalyst supports in chemical industry, was investigated in this work. The specimens are spherical in shape, with a diameter of 5.1 ± 0.2 mm and a pellet density of 1.382 g/cm^3. They are brittle materials fabricated deliberately to produce high porosity and optimized pore distribution.

The tensile strength was measured using a diametral compression test. To adequately characterize the strength properties of the population, a sufficiently large number of the strength tests of 500 was performed. A ZQJ-II strength tester made in Dalian, China, was used, described in detail elsewhere.[4,18] The fracture stress was calculated according to the equation of Hiramatsu and Oka:[19]

$$\sigma = \frac{2.8F}{\pi d^2} \qquad (10)$$

where F is the load at fracture and d is the specimen diameter.

4. Results and discussion

4.1. Probability estimator for the unbiased estimate of the Weibull modulus

By Monte Carlo simulation in Section 2, the combination of the α and β-values leading to an unbiased estimate of the Weibull modulus was determined for each sample size, as listed in Table 1. It is clear that α and β both are sensitive to the sample size; however, there is no a distinct relationship existing between α or β and the sample size.

4.2. Comparisons with the commonly-used probability estimators

A Monte Carlo simulation procedure similar to that mentioned in Section 2 was conducted for the commonly-used probability estimators, Eqs. (3a)–(3d). Fig. 2 shows the dependence of the normalized mean values of the estimated Weibull moduli, $\bar{m}/m_{true}$, on the sample size n for five probability estimators investigated. It can be seen that with the use of the probability estimator proposed in this work the normalized mean Weibull moduli are always equal to unity

Table 1
α and β-values in Eq. (5) leading to unbiased estimate

n	α	β
10	0.37	0.24
15	0.54	0.85
20	0.49	0.32
25	0.47	0.13
30	0.53	0.41
35	0.57	0.64
40	0.56	0.52
45	0.51	0.14
50	0.56	0.42

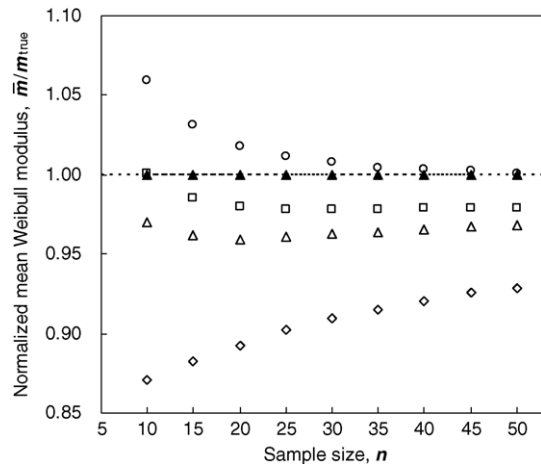Fig. 2. Estimated Weibull modulus as a function of the sample size. (◯) Eq. (3a), (◇) Eq. (3b), (△) Eq. (3c), (□) Eq. (3d), and (▲) unbiased.
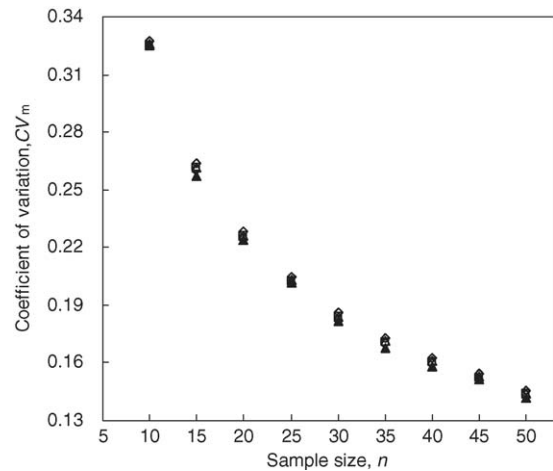

Fig. 3. Dependence of the coefficient of variation of the estimated Weibull modulus on the sample size. (◯) Eq. (3a), (◇) Eq. (3b), (△) Eq. (3c), (□) Eq. (3d), and (▲) unbiased.

at all sample sizes examined, which reveals that this estimator leads to the unbiased estimate of the Weibull modulus. Among the commonly-used probability estimators, Eq. (3a) gives the least-biased estimate of the Weibull modulus for $n \geq 20$. The next is Eq. (3d), followed by Eq. (3c). The estimator, Eq. (3b) leads to the largest bias for all sample sizes examined. It is also clear that as the sample size decreases the bias increases rapidly for Eqs. (3a) and (3d). These results are in agreement with those reported by previous authors.[9–12]

In Fig. 3, the coefficient of variation of the Weibull modulus is plotted as a function of the sample size. Clearly, the coefficient of variation is as expected decreasing with increasing sample size for all probability estimators. It is also shown that the coefficients of variation for the different estimators are approximately equal at all sample sizes; however, it seems the estimator proposed in this work results in the least coefficients of variation at most of the sample sizes.

Statistics textbook tells us that the higher the probability of computing an estimate near to the true value is, the higher the estimation precision is. It indicates that the estimation precision of the Weibull modulus is related not only to the bias but also to the coefficient of variation. The latter describes the dispersion or breadth of the estimated data distribution, while the former shows the centrality or location of the distribution. The smaller both of them are, the more accurate the estimated Weibull modulus is. Based on this criterion, the best probability estimator can be judged from Figs. 2 and 3. Apparently, the probability estimator proposed in this work, leading to the unbiased estimate of the Weibull modulus, gives the most accurate estimation for each sample size, and should, therefore, be preferred. Among the commonly-used probability estimators, Eq. (3b) gives the worst precision of estimation, while Eq. (3a) leads to the highest precision for $n \geq 20$, which was considered as the best probability estimator by many authors.[9–12]

From an engineering point of view, the safety is of the first importance, while the estimation precision is the second.

An overestimation of the Weibull modulus often leads to an underestimation of the probability of failure at low stresses, and hence a lower safety arises in reliability prediction.

Fig. 4 shows the occurrence probability of the Weibull modulus overestimation, i.e. $m/m_{true} > 1$ as a function of the sample size. The higher the probability is, the lower the safety is. It can be seen from Fig. 4 that the probability for the estimator, Eq. (3a) is the highest; however, it is still less than 50% for $n \geq 15$. It implies that for the LR method with different probability estimators, the underestimation of the Weibull modulus always occurs more frequently than the overestimation. It is also clear that the estimator for the unbiased estimate of the Weibull modulus results in a higher safety than Eq. (3a). However, the estimator leading to the highest safety is Eq. (3b) though it gives the worst precision of estimation.
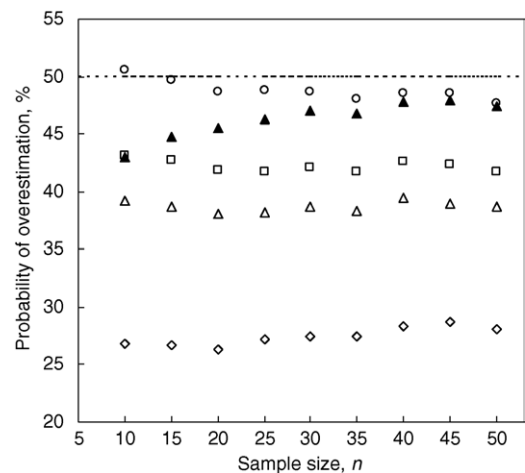

Fig. 4. Occurrence probability of the Weibull modulus overestimated as a function of the sample size. (◯) Eq. (3a), (◇) Eq. (3b), (△) Eq. (3c), (□) Eq. (3d), and (▲) unbiased.

Table 2
Estimated Weibull parameters of experimental data

Methods	m	σ_0 (N/cm^2)
LR, Eq. (3a)	4.42	168.35
LR, Eq. (3b)	4.36	168.53
LR, Eq. (3c)	4.39	168.43
LR, Eq. (3d)	4.40	168.40
ML	4.43	168.23

4.3. Estimation of the scale parameter

Similar Monte Carlo simulations were conducted to produce 10,000 estimated values of the scale parameter, σ_0, for each probability estimator and for each sample size. The mean value and the coefficient of variation were calculated. And then the 10,000 estimated σ_0 were ranked in ascending order and divided into 40 equally sized intervals. The number of estimated σ_0 falling into each interval was counted. This number, normalized through division by 10,000, the total number of estimated σ_0, produces the relative frequency of occurrence, which was taken as the y-value. The midpoint of the given interval was used as the x-value. The resulting histogram can be regarded as an empirical probability density distribution.[5] For the sake of comparison, the same data processing was also carried out for the estimated Weibull moduli.

As an example, the probability density distributions of the estimated Weibull modulus and scale parameter at a sample size of 20 are shown in Fig. 5, where the x-axes of two sub-figures have the same range and the same scale. The results of other sample sizes are similar to those shown in Fig. 5. It can be seen that all probability estimators give a similar distribution of $\sigma_0/\sigma_{0,\text{true}}$ that scatters in the vicinity of the true value in a much smaller range, as compared with the Weibull modulus. In reality, for any sample size the coefficient of variation of the estimated scale parameter is about one tenth of that of the estimated Weibull modulus; therefore, the scale parameter can be estimated with accuracy about an order of magnitude higher than the Weibull modulus.[5,14] From Fig. 5, it is also clear that for the estimation of the scale parameter, there is no significant difference between the probability estimator proposed in this work and the commonly-used ones.

Note that the distribution of m/m_{true} is asymmetrical and lightly skewed to the right; the distribution of $\sigma_0/\sigma_{0,\text{true}}$, however, takes an approximately symmetrical form. Similar results have also been reported by Khalili and Kromp.[5] For any sample size and any probability estimator, the mean value of the estimated scale parameter is always close to its true value, and its bias is negligible. Therefore, the estimator of the scale parameter obtained with any probability estimator is approximately unbiased.

4.4. Experimental validation

Table 2 gives the Weibull parameters of the experimental data, estimated with different methods. In principle, the
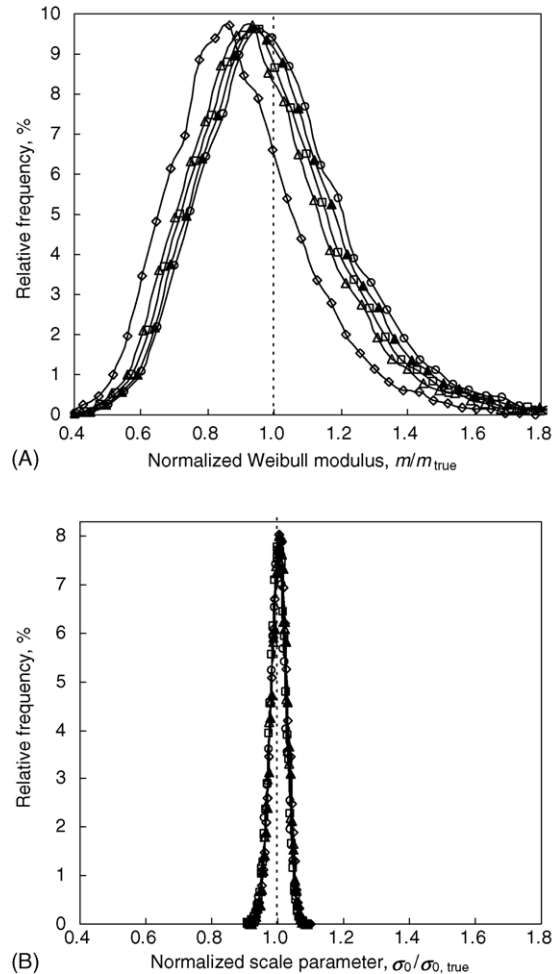

Fig. 5. Probability density distributions of the estimated Weibull modulus (A) and scale parameter (B) at a sample size $n = 20$. (○) Eq. (3a), (◇) Eq. (3b), (△) Eq. (3c), (□) Eq. (3d), and (▲) unbiased.

true values of the Weibull parameters can be obtained definitely only for an infinite number of specimens. However, a sufficiently large number of tested specimens will give a quite good approximation. Especially, the Weibull parameters estimated with the maximum likelihood method may be regarded approximately as the true values.

The advantage of the unbiased estimate is that the mean Weibull modulus from a number of sets of strength data should be close to its true value, which can be verified by the following procedure. The 500 experimentally measured fracture stresses were treated as a full set. To investigate the effects of the sample size, the subsets of the full set were selected in groups of 10, 15, 20, 25, 30, 35, 40, 45 and 50. The subsets were selected randomly from the full set by assigning a computer-generated random number to each member of the full set, sorting the stress data according to that number, and then choosing the first 10, 15, etc., results in the full set. Each subset was considered as an independent set of data, and its Weibull modulus was estimated with the LR method with the estimators, Eqs. (3a)–(3d), and the estimator proposed in
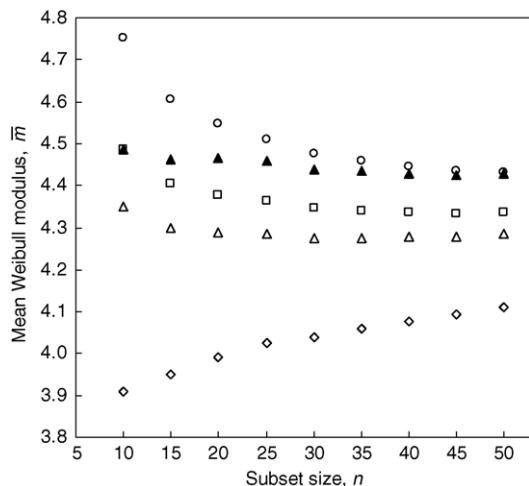
Fig. 6. Weibull modulus of experimental data as a function of the sample size. (◯) Eq. (3a), (◇) Eq. (3b), (△) Eq. (3c), (□) Eq. (3d), and (▲) unbiased.

this work. By repeating this procedure, the total number of the subsets for each sample size amounts to 10,000. Finally, the mean value and the coefficient of variation of the 10,000 Weibull moduli were calculated for each sample size and for each probability estimator.

Fig. 6 shows the dependence of the mean value of the estimated Weibull modulus of actual experimental data on the sample size. Note that the only probability estimator which has leveled off as a function of the subset size is the one proposed in this work. And this estimator yields a Weibull modulus, which is very close to the true value approximated by the ML method. Thus, it can be seen that the probability estimator proposed in this work is certain to give the unbiased estimate of the Weibull modulus. However, the estimators of the Weibull modulus, obtained with the commonly-used probability estimators, are always biased.
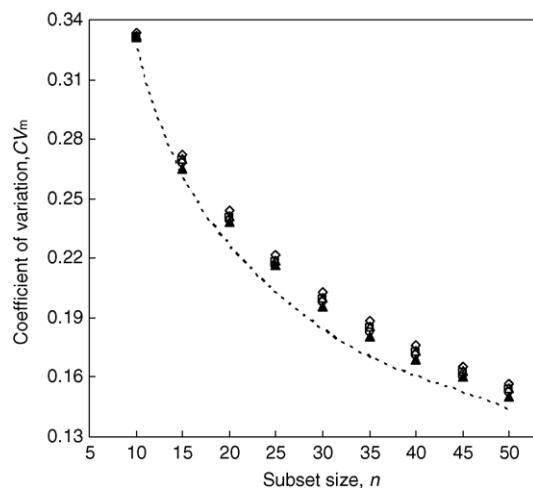

Fig. 7. Coefficient of variation of the Weibull modulus of experimental data vs. the sample size. (◯) Eq. (3a), (◇) Eq. (3b), (△) Eq. (3c), (□) Eq. (3d), and (▲) unbiased.

In Fig. 7 the coefficient of variation of the estimated Weibull modulus of experimental data is plotted as a function of the sample size, where the dotted line is the curve fitting result of the data points shown in Fig. 3. It can be seen that the coefficient of variation of the Weibull modulus decreases with increasing sample size, but independent of the probability estimators, as similar to the results of Monte Carlo simulation. It is also clear that the coefficient of variation from actual experimental data is always higher than that from Monte Carlo simulation at any sample size. The reason for this phenomenon is that the strength data from actual experiments does not perfectly follow the Weibull statistics, which aggravates the dispersion of the estimated Weibull modulus of actual experimental data.

5. Conclusions

Using a Monte Carlo simulation, a probability estimator for the unbiased estimate of the Weibull modulus was determined. Compared with the commonly-used probability estimators, the estimator proposed gives a more accurate estimation of the Weibull modulus, and the same estimation precision of the scale parameter, which is much higher than that of the Weibull modulus. It is concluded that the estimated scale parameter is always approximately unbiased.

The probability estimators were also compared from an engineering point of view. It is found that the probability estimator proposed results in a higher safety than Eq. (3a). However, the estimator leading to the highest safety is Eq. (3b) though it gives the worst precision of estimation.

It is shown that the estimated Weibull modulus from actual experimental data is more dispersive than that from Monte Carlo simulation, which arises from the fact that the strength data from actual experiments does not perfectly follow the Weibull statistics. Finally, the unbiased property of the estimated Weibull modulus was validated with actual experimental data. It is, therefore, recommended that the probability estimator proposed, leading the unbiased estimate of the Weibull parameters, should be used in the linear regression method for estimating the Weibull parameters.

Acknowledgements

References

1. *ASTM standard, C1239-00 Standard practice for reporting uniaxial strength data and estimating Weibull distribution parameters for advanced ceramics*. American Society for Testing and Materials, Philadelphia, PA, 2003.

2. Quinn, G. D., Flexure strength of advanced structural ceramics: a round robin. *J. Am. Ceram. Soc.*, 1990, **73**, 2374–2384.

3. Madjoubi, M. A., Bousbaa, C., Hamidouche, M. and Bouaouadja, N., Weibull statistical analysis of the mechanical strength of a glass eroded by sand blasting. *J. Eur. Ceram. Soc.*, 1999, **19**, 2957–2962.

4. Wu, D. F., Li, Y. D., Shi, Y. H., Fang, Z. P., Wu, D. H. and Chang, L., Effects of the calcination conditions on the mechanical properties of a PCoMo/Al_2O_3 hydrotreating catalyst. *Chem. Eng. Sci.*, 2002, **57**, 3495–3504.

5. Khalili, A. and Kromp, K., Statistical properties of Weibull estimators. *J. Mater. Sci.*, 1991, **26**, 6741–6752.

6. Trustrum, K. and Jayatilaka, A. De S., On estimating the Weibull modulus for a brittle material. *J. Mater. Sci.*, 1979, **14**, 1080–1084.

7. Langlois, R., Estimation of Weibull parameters. *J. Mater. Sci. Lett.*, 1991, **10**, 1049–1051.

8. Song, L. Y., Wu, D. F. and Li, Y. D., Optimal probability estimators for determining Weibull parameters. *J. Mater. Sci. Lett.*, 2003, **22**, 1651–1653.

9. Bergman, B., On the estimation of the Weibull modulus. *J. Mater. Sci. Lett.*, 1984, **3**, 689–692.

10. Wu, D. F., Li, Y. D., Zhang, J. P., Chang, L., Wu, D. H., Fang, Z. P. *et al.*, Effects of the number of testing specimens and the estimation methods on the Weibull parameters of solid catalysts. *Chem. Eng. Sci.*, 2001, **56**, 7035–7044.

11. Sullivan, J. D. and Lauzon, P. H., Experimental probability estimators for Weibull plots. *J. Mater. Sci. Lett.*, 1986, **5**, 1245–1247.

12. Papargyris, A. D., Estimator type and population size for estimating the Weibull modulus in ceramics. *J. Eur. Ceram. Soc.*, 1998, **18**, 451–455.

13. Wu, D. F. and Jiang, H., Comment on A new probability index for estimating Weibull modulus for ceramics with the least-square method. *J. Mater. Sci. Lett.*, 2003, **22**, 1745–1746.

14. Wu, D. F., Lu, G. Z., Jiang, H. and Li, Y. D., Improved estimation of Weibull parameters with the linear regression method. *J. Am. Ceram. Soc.*, 2004, **87**, 1799–1802.

15. Davies, I. J., Empirical correction factor for the best estimate of Weibull modulus obtained using linear least squares analysis. *J. Mater. Sci. Lett.*, 2001, **20**, 997–999.

16. Davies, I. J., Best estimate of Weibull modulus obtained using linear least squares analysis: An improved empirical correction factor. *J. Mater. Sci.*, 2004, **39**, 1441–1444.

17. Peterlik, H., The validity of Weibull estimators. *J. Mater. Sci.*, 1995, **30**, 1972–1976.

18. Li, Y. D., Wu, D. F., Zhang, J. P., Chang, L., Wu, D. H., Fang, Z. P. *et al.*, Measurement and statistics of single pellet mechanical strength of differently shaped catalysts. *Powder Technol.*, 2000, **113**, 176–184.

19. Hiramatsu, Y. and Oka, Y., Determination of the tensile strength of rock by a compression test of an irregular test piece. *Int. J. Rock Mech. Miner. Sci.*, 1966, **3**, 89–99.